Using various methods to solve the equation of tangents to a circle

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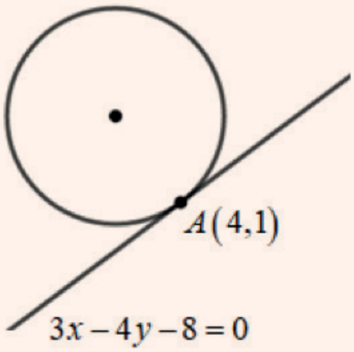
It is important that students can solve mathematics problems by using different methods, which means students understand the connections greatly.

The interesting task shows a tangent equation of a circle and a tangency point. We need to find the unknown equation of a circle. In this essay, we will try to solve this problem in seven ways, such as using discriminant, vector, Pythagorean theorem etc.

Question: The line $3x-4y-8=0$ is tangent to the circle $x^2+y^2-6x+Ey+F=0$ at the point $A(4,1)$. Find the values of E and F.

Method 1 — By discriminant.



Solution: Putting $A(4,1)$ into equation

$$x^2+y^2-6x+Ey+F=0 \quad \text{— (1)},$$

$$4^2+1^2-6\cdot 4+E\cdot 1+F=0$$

$$F=-E+7 \quad \text{— (2)},$$

Putting (2) into (1),

$$x^2+y^2-6x+Ey-E+7=0$$

Finding the intersection of tangent and circle,

$$\begin{cases} 3x-4y-8=0-(3) \\ x^2+y^2-6x+Ey-E+7=0-(4) \end{cases}$$

By (3), $x=\frac{4y+8}{3}$ then putting it into (4),

$$\left(\frac{4y+8}{3}\right)^2+y^2-6\cdot\frac{4y+8}{3}+Ey-E+7=0$$

$$\frac{16y^2+64y+64}{9}+y^2-8y-16+Ey-E+7=0$$

$$16y^2+64y+64+9y^2-72y-144+9Ey-9E+63=0$$

$$25y^2+(-8+9E)y-17-9E=0$$

Discriminant $\Delta=(-8+9E)^2-4\cdot25\cdot(-17-9E)=81E^2+756E+1764$

The intersection of line and circle is just one point, so $\Delta=0$

$$81E^2+756E+1764=0$$

$$9E^2+84E+196=0$$

$$(3E+14)^2=0$$

$$E=-\frac{14}{3}\ .$$

Putting $E=-\frac{14}{3}$ into (2), hence $F=+\frac{14}{3}+7=\frac{35}{3}$.

Method 2 — By "Radius = distance from the centre to the tangent.

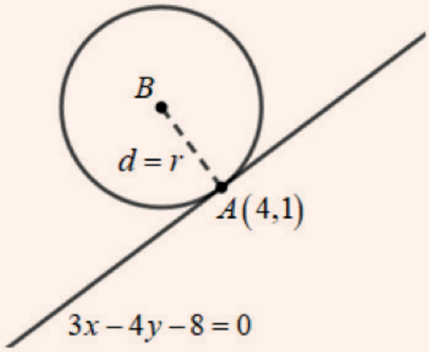
Solution: We can find out the following condition in method 1,

1. $F=-E+7$

2. Equation of circle $x^2+y^2-6x+Ey-E+7=0$

3. The centre of circle B $=\left(3,-\frac{E}{2}\right)$

(It will be considered as a known condition in coming methods.)



According to Radius = distance from the centre to the tangent,

$$\sqrt{(4-3)^2+\left(1+\frac{E}{2}\right)^2}=\frac{\left|3\cdot 3+4\cdot\frac{E}{2}-8\right|}{\sqrt{3^2+4^2}} \quad \text{— (5)}$$

$$\sqrt{1+1+E+\frac{E^2}{4}}=\frac{|1+2E|}{5}$$

$$5\cdot\sqrt{2+E+\frac{E^2}{4}}=|1+2E|$$

$$25\cdot\left(2+E+\frac{E^2}{4}\right)=1+4E+4E^2$$

$$9E^2+84E+196=0$$

$$(3E+14)^2=0$$

$$E=-\frac{14}{3}$$

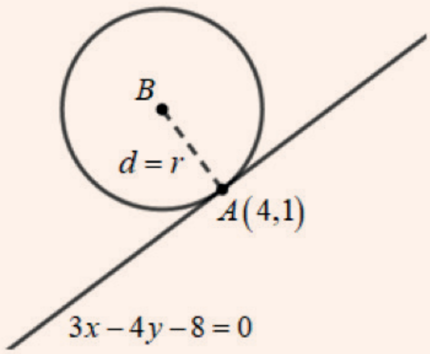
$$F=\frac{35}{3}.$$

Furthermore, we can use the formulae of radiu $r=\frac{1}{2}\sqrt{D^2+E^2-4F}$ instead of the left-hand side of (5).

Remark: In method 1, the intersection of the circle and tangent is a point only. Hence, the discriminant equals to 0. Although the concept is simple, the calculation process is complicated. Therefore, it is quite hard for student to investigate.

Method 3 — By radius is perpendicular to the tangent of the circle.

Solution:



The slope of tangent is $\frac{3}{4}$ and *AB* is perpendicular to the tangent, so the slope of $AB = -\frac{4}{3}$

By the point-slope form, the equation of line *AB* is $4x+3y-19=0$

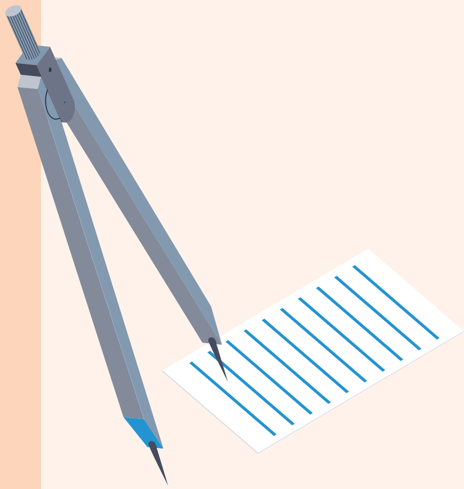
Point $B\left(3,-\frac{E}{2}\right)$ is on the *AB* so that it fulfils the equation of $4x+3y-19=0$.

$$4\cdot 3+3\cdot\left(-\frac{E}{2}\right)-19=0$$

$$-\frac{3E}{2}=7$$

$$E=-\frac{14}{3}$$

$$F=\frac{35}{3}.$$



Method 4 —By the slope of the line joining radius and tangent line.

Solution: By the formula of slope, the slope of *AB* is

$k_{AB}=\dfrac{1+\frac{E}{2}}{4-3}=1+\dfrac{E}{2}$ and the slope of tangent is $k_l=\dfrac{3}{4}$.

Since line *AB* and the tangent are mutually perpendicular lines, $k_{AB}\cdot k_l=-1$.

$$\left(1+\frac{E}{2}\right)\cdot\frac{3}{4}=-1$$

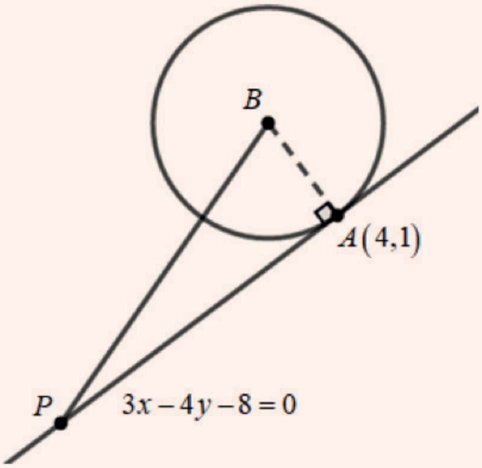
$$E=-\frac{14}{3}$$

$$F=\frac{35}{3}.$$

Remark: The Method 3 is similar to Method 4 which is using the slope of line to formulate the equation and the calculation is easier than both method 1 and 2. Since the above calculation involves linear equation, it can be considered as a better method.

Method 5 —By Pythagorean theorem

Solution:



Taking a point $P(0,-2)$ on the line $3x-4y-8=0$ and connect with $A(4,1)$ and $B\left(3,-\frac{E}{2}\right)$ we have

$$PA=\sqrt{(4-0)^2+(1+2)^2}=5$$

$$PB=\sqrt{3^2+\left(-\frac{E}{2}+2\right)^2}=\sqrt{\frac{E^2}{4}-2E+13}$$

$$AB=\sqrt{(4-3)^2+\left(1+\frac{E}{2}\right)^2}=\sqrt{\frac{E^2}{4}+E+2}$$

And $$AB\perp PA$$

By Pythagorean theorem, $$PB^2=PA^2+AB^2$$

$$\frac{E^2}{4}-2E+13=25+\frac{E^2}{4}+E+2$$

$$3E=-14$$

$$E=-\frac{14}{3}$$

$$F=\frac{35}{3}.$$

Method 6 — By vertical vectors.

Solution: Taking the point $P(0,-2)$ on the tangent. $\overrightarrow{AB}=\left(-1,-\frac{E}{2}-1\right)$, $\overrightarrow{PA}=(4,3)$.

$\overrightarrow{AB}\perp\overrightarrow{PA}$, so $\overrightarrow{AB}\cdot\overrightarrow{PA}=0$

$$-1\cdot 4+3\cdot\left(-\frac{E}{2}-1\right)=0$$

$$E=-\frac{14}{3}$$

$$F=\frac{35}{3}.$$

Method 7 — By normal vector of equation of line.

Solution: Let $\vec{n}$ be the normal vector of tangent, $\vec{n}=(3,-4)$, $\overrightarrow{AB}=\left(-1,-\frac{E}{2}-1\right)$

Vectors $\vec{n}$ and $\overrightarrow{AB}$ are collinear, $\vec{n}=\lambda\overrightarrow{AB}$ where λ is a constant.

$$(3,-4)=\lambda\left(-1,-\frac{E}{2}-1\right)$$

$$\lambda=-3 \text{ and } -4=\lambda\cdot\left(-\frac{E}{2}-1\right)$$

$$-4=-3\cdot\left(-\frac{E}{2}-1\right)$$

$$\frac{4}{3}=-\frac{E}{2}-1$$

$$\frac{E}{2}=-\frac{7}{3}$$

$$E=-\frac{14}{3}$$

$$F=\frac{35}{3}.$$

Remark：The method 5 is not a traditional solution. In method 6 and 7, we solve the problem by vector. The concept is similar to method 3 and 4. However, the expression is shown in other way.

We provide various views and methods to solve the question. Despite the fact that some solutions are not "good", all ideas are worthy. We believe that thinking in different ways is critical for instructors and their students.

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